Anomalous power law distribution of total lifetimes of branching processes: Application to earthquake aftershock sequences

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We consider a general stochastic branching process, which is relevant to earthquakes, and study the distributions of global lifetimes of the branching processes. In the earthquake context, this amounts to the distribution of the total durations of aftershock sequences including aftershocks of arbitrary generation number. Our results extend previous results on the distribution of the total number of offspring (direct and indirect aftershocks in seismicity) and of the total number of generations before extinction. We consider a branching model of triggered seismicity, the epidemic-type aftershock sequence model, which assumes that each earthquake can trigger other earthquakes ("aftershocks"). An aftershock sequence results in this model from the cascade of aftershocks of each past earthquake. Due to the large fluctuations of the number of aftershocks triggered directly by any earthquake ("productivity" or "fertility"), there is a large variability of the total number of aftershocks from one sequence to another, for the same mainshock magnitude. We study the regime where the distribution of fertilities μ is characterized by a power law $\sim 1/\mu^{1+\gamma}$ and the bare Omori law for the memory of previous triggering mothers decays slowly as $\sim 1/t^{1+\theta}$, with $0<\theta<1$ relevant for earthquakes. Using the tool of generating probability functions and a quasistatic approximation which is shown to be exact asymptotically for large durations, we show that the density distribution of total aftershock lifetimes scales as $\sim 1/t^{1+\theta/\gamma}$ when the average branching ratio is critical $(n=1)$. The coefficient $1<\gamma=b/\alpha<2$ quantifies the interplay between the exponent $b\approx 1$ of the Gutenberg-Richter magnitude distribution $\sim 10^{-bm}$ and the increase $\sim 10^{\alpha m}$ of the number of aftershocks with mainshock magnitude m (productivity), with $0.5<\alpha<1$. The renormalization of the bare Omori decay law $\sim 1/t^{1+\theta}$ into $\sim 1/t^{1+\theta/\gamma}$ stems from the nonlinear amplification due to the heavy-tailed distribution of fertilities and the critical nature of the branching cascade process. In the subcritical case $n<1$, the crossover from $\sim 1/t^{1+\theta/\gamma}$ at early times to $\sim 1/t^{1+\theta}$ at longer times is described. More generally, our results apply to any stochastic branching process with a power-law distribution of offspring per parent and a long memory.

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I. INTRODUCTION

We study the distribution of the total duration of an aftershock sequence, for a class of branching processes [1,2] appropriate in particular for modeling earthquake aftershock sequences. The noteworthy particularity and challenging property of this class of branching processes is that the variance of the number of progenies in direct lineage from the mother is mathematically infinite. In addition, a long-time (power-law) memory of the impact of a mother on the first-generation daughters gives rise to subdiffusion [3,4] and to non-mean-field behavior in the distributions of the total number of aftershocks per mainshock and of the total number of generations before extinction [5]. Here, we add to this previous work by showing that the distribution of the total duration of an aftershock sequence is extremely long tailed: the very heavy-tailed nature of the distribution of the durations of aftershock sequences predicted by this simple model may explain the large variability of the lifetimes of observed aftershock sequences, and is compatible with the observation that felt aftershocks of the great Mino-Owari (1891) Japanese earthquake, which inspired Omori's statistical rate model, have persisted at a rate consistent with the Omori law for 100 years [6].

Our results may also be of interest to other systems which are characterized by branching processes with a broad power-law distribution of fertilities, such as epidemic transmission of diseases, and more generally transmission processes involving avalanches spreading on networks such as the World Wide Web, cellular metabolic networks, ecological food webs, social networks, and so on, as a consequence of the well-documented power-law distribution of connectivities among nodes. Our results are thus relevant to systems in which the number of offspring may be large due to long-range interactions, long-memory effects, or large deviation processes.

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II. THE EPIDEMIC-TYPE AFTERSHOCK SEQUENCE BRANCHING MODEL OF EARTHQUAKES WITH LONG MEMORY

We consider a general branching process in which each progenitor or mother (mainshock) is characterized by its conditional average number

$$N_m \equiv \kappa\mu(m) \tag{1}$$

of children (triggered events or aftershocks of the first generation), where

$$\mu(m) = 10^{\alpha(m-m_0)} \tag{2}$$

is a mark associated with an earthquake of magnitude $m \geqslant m_0$ (in the language of "marked point processes"), κ is a constant factor, and m_0 is the minimum magnitude of earthquakes capable of triggering other earthquakes. The meaning of the term "conditional average" for N_m is the following: for a given earthquake of magnitude m and therefore of mark $\mu(m)$, the number r of its daughters of the first generation is drawn at random according to the Poissonian statistics

$$p_\mu(r) = \frac{N_m^r}{r!}e^{-N_m} = \frac{(\kappa\mu)^r}{r!}e^{-\kappa\mu}. \tag{3}$$

Thus, N_m is the expectation of the number of daughters of the first generation, conditioned on a fixed magnitude m and mark $\mu(m)$. The expression (2) for $\mu(m)$ is chosen in such a way that it reproduces the empirical dependence of the average number of aftershocks triggered directly by an earthquake of magnitude m (see [7] and references therein). Expression (1) with (2) gives the so-called productivity law of a given mother as a function of its magnitude.

In addition, we use the well-known Gutenberg-Richter (GR) density distribution of earthquake magnitudes

$$p(m) = b\ln(10)10^{-b(m-m_0)}, \quad m \geqslant m_0, \tag{4}$$

such that $\int_m^\infty p(x)dx$ gives the probability that an earthquake has a magnitude equal to or larger than m. This magnitude distribution $p(m)$ is assumed to be independent of the magnitude of the triggering earthquake, i.e., a large earthquake can be triggered by a smaller one [7,8].

Combining Eqs. (4) and (2), we see that the earthquake marks μ and therefore the conditional average number N_m of daughters of the first generation are distributed according to a power law

$$p_\mu(\mu) = \frac{\gamma}{\mu^{1+\gamma}}, \quad 1 \leqslant \mu < +\infty, \quad \gamma = b/\alpha. \tag{5}$$

Note that $p_\mu(\mu)$ is normalized: $\int_1^{+\infty} d\mu\, p_\mu(\mu)=1$. For earthquakes, $b\approx 1$ almost universally and $0.5<\alpha<1$ giving $1<\gamma<2$ (Ref. [7] reports some evidence for $\alpha\approx 0.8$ corresponding to $\gamma\approx 1.25$ while Ref. [9] argue that $\alpha\approx 1$ leading to $\gamma\to 1$). The fact that $1<\gamma<2$ implies that the mathematical expectation of μ and therefore of N_m (performed over all possible magnitudes) is finite but its variance is infinite (we do not address here the marginal case $\alpha=1$ leading to $\gamma=1$, which needs a special treatment).

For a fixed γ, the coefficient κ then controls the value of the average number n of children of the first generation per mother:

$$n = \langle N_m\rangle = \kappa\langle\mu\rangle = \kappa\frac{\gamma}{\gamma-1}, \tag{6}$$

where the average $\langle N_m\rangle$ is taken over all mothers' magnitudes drawn from the GR law. In the terminology of branching processes, n is called the branching ratio. For $n<1$, there is on average less than one child per mother: this corresponds to transient (subcritical) branching processes with finite lifetimes with probability 1. For $n>1$, there is more than one child per mother: this corresponds to explosive (supercritical) branching processes with a number of events growing exponentially with time. The value $n=1$ of exactly one child per mother on average is the critical point separating the two regimes.

Finally, we assume that a given event (the "mother") of magnitude $m\geqslant m_0$ occurring at time t_i gives birth to other events ("daughters") of the first generation in the time interval between t and $t+dt$ at the rate

$$\phi_\mu(t) = N_m\Phi(t-t_i) = N_m\frac{\theta c^\theta}{(t+c)^{1+\theta}}H(t) \tag{7}$$

where $0<\theta<1$, $H(t)$ is the Heaviside function, c is a regularizing time scale that ensures that the seismicity rate remains finite close to the mainshock and N_m is given by Eq. (1). The time decay rate (7) is called the "direct Omori law" [12,13]. Due to the process of cascades of triggering by which a mother triggers daughters which then trigger their own daughters and so on, the direct Omori law (7) is renormalized into a "dressed" or "renormalized" Omori law [12,13], which is the one observed empirically.

Expressions (1), (2), (4), and (7) define the epidemic-type aftershock sequence (ETAS) model of triggered seismicity introduced by Ogata in the present form [10] and by Kagan and Knopoff in a slightly different form [11].

III. GENERAL FORMALISM IN TERMS OF GENERATING FUNCTIONS

Since we are interested in characterizing the distribution of the random times at which an aftershock sequence triggered by a given mainshock terminates, we take the time of the mainshock of magnitude m at the origin $t=0$ and we do not consider the effect of earlier earthquakes. This is warranted by the fact that sequences of earthquakes generated by different mainshocks are independent in the ETAS branching model.

A. First generation aftershocks

Let us first discuss a more detailed statistical description of first generation aftershocks. Each aftershock arising independently of another preceding aftershock itself born at the random time t_i has a birth time possessing the probability density function (PDF) $\Phi(t-t_i)$ defined in Eq. (7) and the cumulative distribution function (CDF) $b(t)=\int_0^t\Phi(t')dt'$.

Here and everywhere below, the dimensionless time t/c is used, and we replace t by t/c, with the understanding that t or τ means t/c when needed. It is convenient to introduce the complementary CDF of first generation aftershocks

$$a(t) = 1 - b(t) = \frac{1}{(t+1)^{\theta}}. \quad (8)$$

Let us consider a mainshock with mark μ that triggers exactly r aftershocks of the first generation arising at the moments $(t_1, t_2, \ldots, t_r)$. Then the CDF of the time $T(\mu|r)$ of the last arising aftershock is equal to

$$P_{\mu}(t|r) = \Pr[T(\mu|r) = \max\{t_1, t_2, \ldots, t_r\} < t] = [b(t)]^r. \quad (9)$$

Averaging this CDF over the random first-generation aftershock numbers r at fixed μ weighted by their probability $p_{\mu}(r)$ given by Eq. (3) yields the CDF $P_{\mu}(t)$ for the total duration $T(\mu)$ of the first-generation aftershocks:

$$P_{\mu}(t) = \Pr[T(\mu) < t] = G_{\mu}[b(t)]. \quad (10)$$

Here, $G_{\mu}(z) = \sum_{r=0}^{\infty} p_{\mu}(r) z^r$ is the generating probability function (GPF) of the number of first-generation aftershocks. For the Poissonian statistics (3), it is equal to

$$G_{\mu}(z) = e^{\kappa\mu(z-1)}. \quad (11)$$

This leads to the well-known relation

$$P_{\mu}(t) = e^{-\kappa\mu a(t)}. \quad (12)$$

In the ETAS model, the Gutenberg-Richter distribution (4) of magnitudes together with the productivity law (2) implies the power law (5) for the marks μ. Averaging over all possible mainshock magnitudes thus amounts to averaging Eq. (10) over all possible μ's. The CDF of durations T of first-generation aftershocks generated by some mother of arbitrary magnitude arising at time $t=0$ is equal to

$$P(t) = G[b(t)], \quad (13)$$

where $G(z) = \langle G_{\mu}(z) \rangle$ is the average of $G_{\mu}[b(t)]$ over the random magnitudes m (or, equivalently, random marks μ). In the relevant case of the Poissonian GPF (11) and using (5), we obtain

$$G(z) = \gamma\kappa^{\gamma}(1-z)^{\gamma}\Gamma(-\gamma, \kappa(1-z)), \quad (14)$$

where $\Gamma(x,y)$ is the incomplete Gamma function and $\gamma = b/\alpha$. For real aftershocks, $1 < \gamma < 2$ and a typical value is $\gamma \approx 1.25$. Then, it is easy to show that the first terms of $G(z)$ in a power expansion with respect to $1-z$ are

$$G(z) \simeq 1 - n(1-z) + \beta(1-z)^{\gamma}, \quad 1 < \gamma < 2, \quad (15)$$

with n given by Eq. (6) and

$$\beta = n^{\gamma}\left(\frac{\gamma-1}{\gamma}\right)^{\gamma}\frac{\Gamma(2-\gamma)}{\gamma-1}. \quad (16)$$

B. All generation aftershocks

In the ETAS model, any event (the initial mother or any aftershock, whatever its generation number) triggers its aftershocks of the first generation in a statistically independent and equivalent manner, according to the laws given in Sec. II. This gives the possibility of obtaining closed equations for the CDF of the total duration of aftershock triggering processes.

Let $\mathcal{T}$ be the random waiting time between a mainshock and one of its first-generation aftershocks, chosen arbitrarily. The PDF of $\mathcal{T}$ is nothing but $\Phi(t)$ defined in Eq. (7). Let $\mathbb{T}$ be the random duration of the aftershock branching process triggered by this first-generation aftershock. The CDF of $\mathbb{T}$ is denoted $\mathbb{P}(t)$. Then the total duration, measured since the mainshock, of the sequence of aftershocks generated by this particular first-generation aftershock is $\mathcal{T}+\mathbb{T}$. The CDF $\mathbb{F}(t)$ of this sum is therefore the convolution

$$\mathbb{F}(t) = \Phi(t) \otimes \mathbb{P}(t). \quad (17)$$

Replacing $b(t)$ in Eq. (10) by $\mathbb{F}(t)$ and taking into account the equality (11), we obtain the CDF of the total duration $\mathbb{T}(\mu)$ of a sequence of aftershocks over all generations of a given event of mark μ that occurred at $t=0$:

$$\mathbb{P}_{\mu}(t) = \Pr[\mathbb{T}(\mu) < t] = e^{-\kappa\mu\mathbb{R}(t)}, \quad (18)$$

where

$$\mathbb{R}(t) = 1 - \mathbb{F}(t) \quad (19)$$

is the distribution complementary to the $\mathbb{F}(t)$ CDF defined in Eq. (17). Correspondingly, replacing P(t) in Eq. (13) by $\mathbb{P}(t)$ and $b(t)$ by $\mathbb{F}(t)$, we obtain the self-consistent equation for the CDF $\mathbb{F}(t)$

$$\mathbb{P}(t) = G[\mathbb{F}(t)] = G[\Phi(t) \otimes \mathbb{P}(t)]. \quad (20)$$

It is convenient to rewrite Eq. (20) as

$$\mathbb{R}(t) - \mathbb{Q}(t) = \Omega[\mathbb{R}(t)], \quad (21)$$

where $\mathbb{Q}(t) = 1 - \mathbb{P}(t)$ and

$$\Omega(z) = G(1-z) + z - 1. \quad (22)$$

For our subsequent analysis, expression (21) is more convenient than Eq. (20) for the following reasons. First of all, instead of the CDF's $\mathbb{P}(t)$ and $\mathbb{F}(t)$ entering in Eq. (20), Eq. (21) is expressed in terms of the complementary CDF's $\mathbb{Q}(t)$ and $\mathbb{R}(t)$, which both tend to zero for $t \to \infty$. In addition, the function $\Omega(z)$ also tends to zero for $z \to 0$. This gives the possibility of extracting the influence of the nonlinear terms of the GPF $G(z)$ on the asymptotic behavior of the solution for $t \to \infty$. Indeed, at least for $\gamma \lesssim 1.5$, the GPF $G(z)$ is very precisely described by the truncated series (15). The corresponding series for $\Omega(z)$ is

$$\Omega(z) \simeq (1-n)z + \beta z^{\gamma}, \quad (23)$$

which reduces to a pure power law in the critical case $n=1$:

$$\Omega(z) \simeq \beta z^{\gamma}. \quad (24)$$

Correspondingly, in the critical case $n=1$ and most important for earthquake applications for which $1 < \gamma < 2$ holds, Eq. (21) has the form

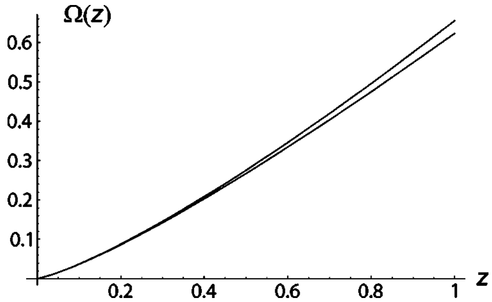

FIG. 1. Plots of the exact function $\Omega(z)$ defined by Eq. (22) (lower curve) and its pure power approximation Eq. (24) (upper curve) for $\gamma=1.25$ and $n=1$.

$$\mathbb{R}(t) - \mathbb{Q}(t) = \beta \mathbb{R}^{\gamma}(t). \tag{25}$$

The exact auxiliary function $\Omega(z)$ defined by Eq. (22) for $n=1$ and its power approximation (24) for $\gamma=1.25$ are shown in Fig. 1.

Our goal is now to solve Eq. (21) and in particular Eq. (25) to explore in detail the statistical properties of the durations of aftershock sequences, resulting from cascades of triggered events.

IV. FRACTIONAL ORDER DIFFERENTIAL EQUATION FOR THE COMPLEMENTARY CDF R(t)

In order to exploit Eq. (21), we first need to express $\mathbb{Q}(t)$ as a function of $\mathbb{R}(t)$. For this, we note that expression (17) is equivalent to

$$\mathbb{R}(t) = a(t) + \Phi(t) \otimes \mathbb{Q}(t), \tag{26}$$

as can be seen from direct substitutions using Eqs. (8) and (19), and $\mathbb{Q}(t)=1-\mathbb{P}(t)$. Applying the Laplace transform to both sides of this equality, one gets

$$\hat{\mathbb{Q}}(s) = \frac{\hat{\mathbb{R}}(s)}{\hat{\Phi}(s)} - \frac{1-\hat{\Phi}(s)}{s\hat{\Phi}(s)}, \tag{27}$$

where

$$\hat{\Phi}(s) = \int_0^{\infty} \Phi(t) e^{-st} dt = \theta (cs)^{\theta} e^{cs} \Gamma(-\theta, cs), \tag{28}$$

where we have made the correspondence $t \to t/c$ explicit [c is defined in Eq. (7)]. We shall be interested in the probability distribution of the durations of total sequences of aftershocks for durations much larger than c. In this case, one can replace $\hat{\Phi}(s)$ by its asymptotics for small s,

$$\hat{\Phi}(s) \simeq 1 - \delta (cs)^{\theta} \simeq \frac{1}{1+\delta (cs)^{\theta}}, \quad cs \ll 1, \tag{29}$$

where $\delta=\Gamma(1-\theta)$. Substituting it into Eq. (27) leads to

$$\hat{\mathbb{Q}}(s) = [1 + \delta (cs)^{\theta}] \hat{\mathbb{R}}(s) - \delta c^{\theta} s^{\theta-1}, \tag{30}$$

which is equivalent, under the inverse Laplace transform, to the fractional order differential equation

$$\mathbb{Q}(t) = \mathbb{R}(t) + \delta c^{\theta} \frac{d^{\theta}\mathbb{R}(t)}{dt^{\theta}} - \left(\frac{c}{t}\right)^{\theta}. \tag{31}$$

Equation (21) thus yields the following fractional order differential equation for $\mathbb{R}(t)$ (going back to the reduced time variable $\tau=t/c$):

$$\delta \frac{d^{\theta}\mathbb{R}}{d\tau^{\theta}} + \Omega(\mathbb{R}) = \tau^{-\theta}. \tag{32}$$

In particular, in the critical case $n=1$, using the power approximation (24), we obtain

$$\delta \frac{d^{\theta}\mathbb{R}}{d\tau^{\theta}} + \beta \mathbb{R}^{\gamma} = \tau^{-\theta}. \tag{33}$$

Note that the nonlinear fractional order differential equation (32) is exact for $\Phi(t)$ given by

$$\Phi(t) = \frac{1}{\delta^{1/\theta}} \Phi_{\theta}\left(\frac{t}{\delta^{1/\theta}}\right), \tag{34}$$

where $\Phi_{\theta}(t)$ is the fractional exponential distribution possessing the Laplace transform

$$\hat{\Phi}_{\theta}(s) = \frac{1}{1+s^{\theta}}, \tag{35}$$

which has the integral representation

$$\Phi_{\theta}(\tau) = \int_0^{\infty} \frac{1}{x} \exp\left(-\frac{\tau}{x}\right) \xi_{\theta}(x) dx, \tag{36}$$

where

$$\xi_{\theta}(x) = \frac{1}{\pi x} \frac{\sin(\pi\theta)}{x^{\theta} + x^{-\theta} + 2\cos(\pi\theta)}. \tag{37}$$

One can interpret Eq. (36) as the decomposition of the fractional exponential law into regular exponential distributions, and $\xi_{\theta}(x)$ given by Eq. (37) as the "spectrum" of their mean characteristic decay times x. For $\theta \to 1$, the spectrum (37) weakly converges to the delta function $\delta(x-1)$ and the fractional exponential law transforms into the regular exponential distribution $\Phi_1(\tau)=e^{-\tau}$. For $\theta=1/2$, there is an explicit expression for the fractional exponential distribution,

$$\Phi_{1/2}(\tau) = \sqrt{\frac{1}{\pi\tau}} - e^{\tau} \operatorname{erfc}(\sqrt{\tau}). \tag{38}$$

It is easy to show that the asymptotics of the fractional exponential distribution are

$$\Phi_{\theta}(\tau) \simeq \frac{\tau^{\theta-1}}{\Gamma(\theta)} \ (\tau \ll 1), \quad \Phi_{\theta}(\tau) \simeq \frac{\theta\tau^{-\theta-1}}{\Gamma(1-\theta)} \ (\tau \gg 1). \tag{39}$$

Figure 2 shows a log-log plot of the Omori law $\Phi(t)$ defined in Eq. (7) and of the corresponding fractional exponential distribution (34) as a function of the reduced time $\tau=t/c$ and for $\theta=1/2$, demonstrating the closeness of these two distributions.

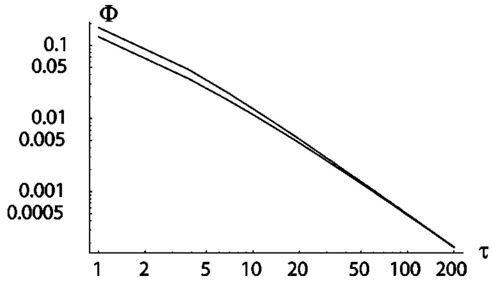

FIG. 2. Log-log plots of the direct Omori law $\Phi(t)$ defined in Eq. (7) (lower curve) and of the fractional exponential distribution Eq. (34) (upper curve) for $\theta=0.5$ and $c=1$.

V. EXACTLY SOLVABLE CASE: PURE EXPONENTIAL OMORI LAW

Before addressing the case of interest for earthquakes where the direct Omori law $\Phi(t)$ is a power law with exponent $0<\theta<1$, it is instructive to present the solution for the case where $\Phi(t)$ is an exponential. In this case, an exact solution can be obtained in closed form. This exact solution will be useful to check the quasistatic and dynamical linearization approximations developed below to solve the difficult case where $\Phi(t)$ is a power law with exponent $0<\theta<1$.

We write the exponential direct Omori law in nonreduced time as

$$\Phi(t)=\frac{1}{c}\exp\left(-\frac{t}{c}\right)\Rightarrow\hat{\Phi}(s)=\frac{1}{1+cs}, \tag{40}$$

so that Eq. (27) transforms to

$$\hat{\mathbb{Q}}(s)=(1+cs)\hat{\mathbb{R}}(s)-c. \tag{41}$$

After taking the inverse Laplace transform, we get

$$\mathbb{Q}(t)=\mathbb{R}(t)+c\frac{d\mathbb{R}(t)}{dt}-c\delta(t), \tag{42}$$

and Eq. (21) takes the form

$$c\frac{d\mathbb{R}(t)}{dt}+\Omega[\mathbb{R}(t)]=c\delta(t), \tag{43}$$

or, in the more traditional form of a Cauchy problem,

$$\frac{d\mathbb{R}}{d\tau}+\Omega[\mathbb{R}]=0, \quad \mathbb{R}(\tau=0)=1. \tag{44}$$

The numerical solution of (44) is easy to obtain. In addition, using for $\Omega(z)$ the series approximation (23), one obtains the analytical solution of the Cauchy problem (44) in the form

$$\mathbb{R}=\left[\left(1+\frac{\beta}{1-n}\right)\exp\left((1-n)\frac{\tau}{g}\right)-\frac{\beta}{1-n}\right]^{-g}, \tag{45}$$

where $g=1/(\gamma-1)$. In particular, in the critical case $n=1$, this leads to

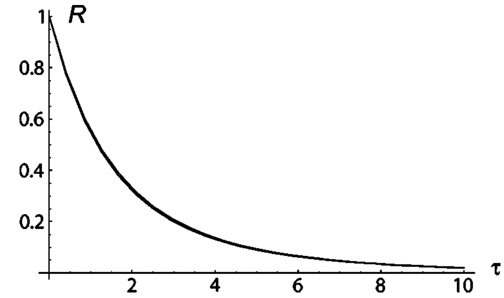

FIG. 3. Plot of the numerical solution of Eq. (44) for the complementary CDF $\mathbb{R}$ of the total duration of an aftershock sequence and the corresponding analytical approximate expression (45) for $\mathbb{R}$ for the parameters $\gamma=1.25$ and $n=0.99$.

$$\mathbb{R}=\left(1+\frac{\beta}{g}\tau\right)^{-g}. \tag{46}$$

Figure 3 shows the numerical solution of Eq. (44) together with its analytical solution (45) obtained using the polynomial approximation (23) of the function $\Omega(z)$ defined in Eq. (22), for $\gamma=1.25$ and $n=0.99$. These two curves are very close to each other.

Note that, in the subcritical case $n<1$, there is a crossover from the power law (46) at early times, which is characteristic of the critical regime $n=1$, to an exponential decay at long times of the complementary CDF $\mathbb{R}$.

VI. DYNAMICAL LINEARIZATION AND QUASISTATIC APPROXIMATIONS TO OBTAIN THE ASYMPTOTIC TAIL OF THE DISTRIBUTION OF TOTAL AFTERSHOCK DURATIONS

A. Linear approximation

To obtain some rough estimate of the complementary CDF $\mathbb{R}(t)$, let us consider the linearized version of the fractional order differential equation (32)

$$\delta\frac{d^{\theta}\mathbb{R}}{d\tau^{\theta}}+\eta\mathbb{R}=\tau^{-\theta}, \tag{47}$$

where the following linearization has been used:

$$\Omega[\mathbb{R}]\simeq\eta\mathbb{R}, \quad \eta=\Omega(1)=G(0). \tag{48}$$

The Laplace transform of the solution of the linearized equation (47) has the form

$$\hat{\mathbb{R}}(s)=\frac{\delta s^{\theta-1}}{\eta+\delta s^{\theta}}. \tag{49}$$

The corresponding complementary CDF is equal to

$$\mathbb{R}=E_{\theta}\left(-\frac{\eta}{\delta}\tau^{\theta}\right), \quad \delta=\Gamma(1-\theta), \tag{50}$$

where $E_{\theta}(z)$ is the Mittag-Leffler function. Its integral representation is

$$E_\theta(-x) = \frac{x}{\pi}\sin\pi\theta\int_0^\infty \frac{y^{\theta-1}e^{-y}dy}{y^{2\theta}+x^2+2xy^\theta\cos\pi\theta}(x>0). \tag{51}$$

In particular, for $\theta=1/2$, it is equal to

$$E_{1/2}(-x) = e^{x^2}\,\mathrm{erfc}(x). \tag{52}$$

Its asymptotics reads

$$E_\theta(-x) \sim \frac{1}{x\delta}\ (x\to\infty), \tag{53}$$

which is already very precise for $x \gtrsim 2$.

The suggested dynamical linearization approach consists in replacing the factor η in Eq. (48) by

$$\eta(\mathbb{R}) = \frac{\Omega(\mathbb{R})}{\mathbb{R}} \tag{54}$$

to correct for the nonlinear decay of the relaxation of the complementary CDF $\mathbb{R}$ as a function of time. It is interesting to check the validity of this dynamical linearization procedure for the exactly solvable exponential Omori law (40). In this case, the solution of the linearized equation (44) is

$$\mathbb{R} = e^{-\eta\tau}. \tag{55}$$

Substituting here Eq. (54) for η, we obtain in the critical case the transcendent equation

$$\mathbb{R} = \exp(-\tau\beta\mathbb{R}^{\gamma-1}). \tag{56}$$

Its solution is equal to

$$\mathbb{R} = \left(\frac{Y(x)}{x}\right)^g, \tag{57}$$

where

$$g = \frac{1}{\gamma-1}, \quad x = \frac{\tau\beta}{g}, \tag{58}$$

and $Y(x)$ is the solution of the transcendent equation $Y\,e^Y = x$. For $x>2$, there is a very precise approximate solution of this equation:

$$Y(x) \simeq \ln x\left[1+(1+\ln x)\left(1-\sqrt{1+\frac{2\ln(\ln x)}{(1+\ln x)^2}}\right)\right] \sim \ln x. \tag{59}$$

Thus, for large x, the main asymptotics of the dynamical linearization approximation (57) of the Cauchy problem (44) differs from the main asymptotics $\mathbb{R}\sim x^{-g}$ of the exact solution (46) only by the logarithmic correction $\ln^g x$.

B. Quasistatic approximation

Close inspection of the complementary CDF (50) and its asymptotics

$$\mathbb{R} \simeq \frac{1}{\eta\tau^\theta}, \quad \tau \gtrsim \tau^*, \quad \tau^* = \left(\frac{2\delta}{\eta}\right)^{1/\theta} \tag{60}$$

derived from relation (53) gives us a hint on how to approach the solution of the nonlinear fractional order differential equations (32) and (33) by using a quasistatic approximation. Indeed, notice that the asymptotics (60) is a solution of the truncated equation (47)

$$\eta\mathbb{R} = \tau^{-\theta}, \tag{61}$$

where we omitted the fractional order derivative term.

Applying this same quasistatic approximation to the nonlinear fractional order differential equation (32) gives the approximate equality

$$\Omega[\mathbb{R}] \simeq \tau^{-\theta}. \tag{62}$$

In particular, in the critical case $n=1$ for which $\Omega(z)\simeq\beta z^\gamma$, we have $\beta\mathbb{R}^\gamma \simeq \tau^{-\theta}$, or equivalently

$$\mathbb{R} \simeq \beta^{-1/\gamma}\tau^{-\theta/\gamma}. \tag{63}$$

Expression (63) will lead to the main result (68) below.

The validity of this quasistatic approximation is checked by calculating the derivation of fractional order θ of the approximate solution (63). Using the standard tabulated formula of fractional order analysis

$$\frac{d^\theta\tau^p}{d\tau^\theta} = \frac{\Gamma(1+p)}{\Gamma(1+p-\theta)}\tau^{p-\theta}, \tag{64}$$

we obtain

$$\delta\frac{d^\theta\mathbb{R}}{d\tau^\theta} \simeq -\beta^{-1/\gamma}\frac{\theta}{\gamma+1}\mathrm{B}\left(-\theta,-\frac{\theta}{\gamma}\right)\tau^{-\theta-\theta/\gamma}, \tag{65}$$

where $\mathrm{B}(x,y)$ is the Beta function. For any fixed $1<\gamma<2$ and $0<\theta<1$, there is a $\tau^*(\gamma,\theta)<\infty$ such that

$$\left|\delta\frac{d^\theta\mathbb{R}}{d\tau^\theta}\right| \ll \beta\mathbb{R}^\gamma \simeq \tau^{-\theta} \quad \text{if } \tau \gg \tau^*(\gamma,\theta) \tag{66}$$

so that the quasistatic approximation becomes applicable. The physical background of the power asymptotics (60) of the solution of the linear equation (47) and of the quasistatic approximation (63) of the nonlinear equation (33) is obvious: the asymptotics $\mathbb{R}\sim\tau^{-\theta}$ given by Eq. (60) is a consequence of the power tail $\Phi(t)\sim t^{-\theta-1}$ of the bare Omori law, while the more slowly decaying $\mathbb{R}\sim\tau^{-\theta/\gamma}$ given by Eq. (63) is the result of an interplay between the long-memory property of the bare Omori law and the amplification by the power law $\Omega(z)\sim z^\gamma$, a signature of the broad distribution of productivities of daughter aftershocks from mother earthquakes. This gives rise to a renormalization of the exponent θ into a smaller exponent θ/γ (for $1<\gamma<2$).

C. PDF of the total duration of aftershock branching processes

The previous sections have discussed in detail how to obtain the complementary CDF $\mathbb{R}$ of the total duration of aftershock branching processes, corresponding to some first-

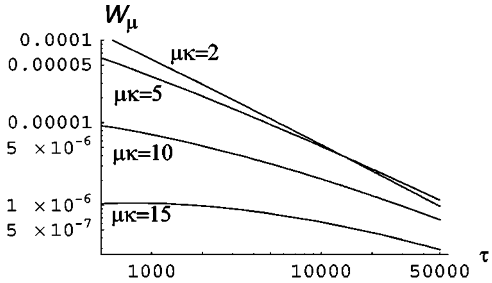

FIG. 4. Log-log plots of the PDF (68) of the total aftershock sequence durations for a mainshock of mark μ, with $\mu\kappa=2,5,10,15$, for the parameter values $\gamma=1.25$, $\theta=0.2$, and $n=1$.

generation aftershock, triggered by a main earthquake. The CDF $\mathbb{P}_\mu$ of the total duration of aftershock triggering processes, taking into account all aftershocks triggered by a main earthquake of fixed magnitude, is described by relation (18). The corresponding PDF of the total duration of an aftershock sequence is thus equal to

$$\mathbb{W}_\mu(\tau) = -\mu\kappa e^{-\kappa\mu\mathbb{R}(\tau)}\frac{d\mathbb{R}(\tau)}{d\tau}. \tag{67}$$

If $\mu\kappa \gg 1$ (as is the case for a large earthquake which has a large average productivity), then, due to the exponential factor in Eq. (67), this PDF differs significantly from zero only if $\mathbb{R}$ is very small. Then using the expression for small values of $\mathbb{R}$ described by the quasistatic approximation (63), we obtain

$$\mathbb{W}_\mu(\tau) = \frac{d\mathbb{P}_\mu(\tau)}{d\tau} \simeq \frac{\theta\mu\kappa}{\gamma\beta^{1/\gamma}}\tau^{-1-\theta/\gamma}\exp\left(-\frac{\mu\kappa}{\beta^{1/\gamma}}\tau^{-\theta/\gamma}\right). \tag{68}$$

Expression (68) is our main result. Figure 4 shows a log-log plot of the PDF (68) for different values of the mainshock size $\mu\kappa$ for $\gamma=1.25$ and $\theta=0.2$ [recall that β is given by Eq. (16) and we put it equal to 1 to draw Fig. 4].

Expression (68) shows that the power-law tail holds for durations $t/c > t_\mu/c \propto (\mu\kappa)^{\gamma/\theta} \sim 10^{(\alpha\gamma/\theta)m}$ for which the exponential factor goes to 1. Thus, for θ small (≈ 0.1–0.3 as seems to be relevant for earthquakes), expression (68) exhibits a very strong dependence on the mainshock magnitude through its impact (2) on the mark μ. Therefore, the most relevant part of the distribution of the durations for small mainshocks is controlled by the power-law tail $\tau^{-1-\theta/\gamma}$. In contrast, the observable part of the distribution of durations for very large mainshocks is controlled by the exponential term which, together with the power-law prefactor, leads to a maximum: for very large μ, $\mathbb{W}_\mu(\tau)$ starts from zero for $\tau=0$ and then increases up to a maximum before crossing over slowly to the power-law tail $\tau^{-1-\theta/\gamma}$, as illustrated in Fig. 4.

D. Crossover from critical to subcritical regime

The asymptotics of the complementary CDF $\mathbb{R}$ satisfies Eq. (62) in the quasistatic approximation. In the subcritical regime, using the polynomial approximation (23), one can rewrite Eq. (62) in the form

$$(1-n)\mathbb{R} + \beta\mathbb{R}^\gamma = \tau^{-\theta}. \tag{69}$$

It is seen from this equality that if $\mathbb{R} > \mathbb{R}_c$, where

$$\mathbb{R}_c = \left(\frac{1-n}{\beta}\right)^g, \tag{70}$$

then one can neglect the linear term in the left-hand side of equality (69) and obtain the power law (63), typical of the critical regime $n=1$. In contrast, if $\mathbb{R} < \mathbb{R}_c$, then the subcritical scenario of the complementary CDF $\mathbb{R}$ dominates and equality (69) gives the subcritical power law

$$\mathbb{R} \simeq \frac{\tau^{-\theta}}{1-n}. \tag{71}$$

It follows from Eqs. (69) and (70) that the time of the crossover from the critical to the subcritical regime is equal to

$$\tau_c \simeq \left(\frac{\beta^g}{(1-n)^{g+1}}\right)^{1/\theta}. \tag{72}$$

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